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Higher dimensional supersymmetry in 4D superspace

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Abstract: We present an explicit formulation of supersymmetric Yang-Mills theories from $\mathcal{D} = 5$ to 10 dimensions in the familiar $\mathcal{N} = 1, \mathcal{D} = 4$ superspace. This provides the rules for globally supersymmetric model building with extra dimensions and in particular allows us to simply write down $\mathcal{N} = 1$ SUSY preserving interactions between bulk fields and fields localized on branes. We present a few applications of the formalism by way of illustration, including supersymmetric "shining" of bulk fields, orbifolds and localization of chiral fermions, anomaly inflow and super-Chern-Simons theories.

Keywords: Field Theories in Higher Dimensions, Superspaces.

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1. Introduction

In recent years there has been a resurgence of interest in theories with extra dimensions, which are in one way or another more accesible than dimensions compactified at the Planck scale. Working with dimensions larger than the Planck length allows us to study higher-dimensional physics in a "bottom-up" approach, within a sensible effective field theory. The new space in extra dimensions has opened up a number of novel approaches to old questions in beyond the standard model physics. Theories where the SM fields are stuck to a 3-brane while gravity is free to propagate in extra dimensions have been used to address the hierarchy problem [1, 2], allowing us to lower the fundamental scale of gravity, and the ultimate cut-off on effective field theory, to TeV energies. This also opens up the possibility that the SM fields can propagate in extra dimensions of a size near the TeV scale [3, 4]. Since the higher-dimensional gauge theory becomes strongly coupled in the UV, it must be embedded in a sensible UV completion not far above a TeV, and this is possible if the fundamental scale is itself in this region.

Many interesting model-building possiblities involve non-gravitational fields propagating in the extra dimensions. For instance, sources for massive bulk fields can "shine" an exponentially falling profile for them in the bulk [5], which can be used to explain small Fermion masses. Another possibility is that different SM Fermions can be localized to different points in the extra dimensions [6]; their small overlapping wavefunctions could also lead to a mass hierarchy, or proton stability. Electroweak symmetry breaking can be triggered by the SM gauge interactions getting strong in extra dimensions [7], and a number of interesting models for SUSY breaking put the SM fields in the bulk [8, 9, 10].

Many of these mechanisms are generic to the existence of extra dimensions and have nothing to do with their size per-se: they could in principle work just as well with extra dimensions near the GUT scale as near the TeV scale. However, if these extra dimensions are to be far above the TeV scale, some physics other than a low fundamental cut-off must be used to stabilize the electroweak scale. Supersymmetry is a natural candidate to do this. Then, in order to be able to work with extra dimensions, we need to know the rules for building supersymmetric theories in higher dimensions. Furthermore, since many of the models use fields localized on 3-branes in the extra dimensions,(for instance on D-branes or at orbifold fixed points), it is also of interest to be able to couple bulk fields to localized fields in a way preserving at least $\mathcal{N}=1$ SUSY in 4D.

It is therefore desirable to have systematic rules for writing down supersymmetric lagrangians in higher dimensions, allowing supersymmetric couplings to fields localized on 3-branes. The main work along these lines we are aware of is the pioneering paper of Peskin and Mirabelli [10], which showed how to couple 5D vector and hyper-multiplets to boundaries in a supersymmetric way, using an off-shell component formalism. However, the formalism is not familiar to 4D SUSY model-builders, and the extension of the formalism to higher dimensions is not obvious.

In this paper, we will present a formalism for explicilty constructing higher-dimensional SUSY theories in a simple way, within the familiar $\mathcal{N}=1, \mathcal{D}=4$ superspace. The simple observation is that, whatever the higher-dimensional theories are, they certainly contain the ordinary 4D SUSY, and therefore they must have an ordinary 4D superspace description. The superfield content of the 4D theory is easy to guess, simply by knowing the total number of SUSY generators in the full theory. For instance in 5D, the smallest spinor is a Dirac spinor with 8 real components, which means there are a minimum of 8 supercharges, or $\mathcal{N}=2$ in 4D. From the 4D viewpoint, we have either hypermultiplets or vector multiplets. Consider hypermultiplets for simplicity. In $\mathcal{N}=1$ language, they break into two chiral multiplets H, H^c. Furthermore, we have one of these superfields for each point x_5 in the 5'th dimension. So, our field content consist of superfields $H(x_5), H^c(x_5)$. From the 4D point of view x_5 can simply be thought of as a label. Now, our task is to write down a superspace action for these fields that, once all auxilliary fields have been integrated out, reduces to the correct component action for the 5D theory. This is very easy to do, as the possible terms are heavily constrained by various symmetries. For this particular example this was done in [11], and will be reviewed in the next section. We will carry this procedure out for all globally supersymmetric theories from $\mathcal{D}=5$ to 10 dimensions in this paper. But in any case, once we have the action for the bulk theory written in 4D superspace, it is trivial to couple bulk fields to fields localized on 3-branes, in a way preserving $\mathcal{N}=1$ SUSY. We simply add additional 4D superspace interactions localized at particular locations in the transverse dimensions.

We will begin by describing SUSY gauge theories in $5,6$ dimensions, where the field content is the same as $\mathcal{N}=2$ in 4D. We then move on to the cases $\mathcal{D}=7$ to 10, where the field content is that of $\mathcal{N}=4$ in 4D. For the gauge multiplets, we first discuss the abelian theory before giving the non-abelian generalizations. We then discuss a number of applications in the remainder of the paper.

After this work was posted to hep-th, we were informed by A. Sagnotti and W. Siegel that a superfield formulation of $\mathcal{D}=10$ SYM was given in [12]. The formulation there is essentially identical to the one we present for this case. The action given in [12] has an extra Wess-Zumino-Witten type term required to make it fully gauge invariant — this term was missed in the first version of our paper. However, the new term vanishes in Wess-Zumino gauge, and so our previous results are unmodified in WZ gauge. [12] did not discuss the construction of minimally supersymmetric models in $\mathcal{D}=5,6$, nor the applications of the formalism to brane-bulk couplings and model-building.

2. $\mathcal{D}=5,6$

In $\mathcal{D}=5$ the smallest spinor is a 4 component Dirac spinor with 8 real degrees of freedom. In $\mathcal{D}=6$ the smallest spinor is a 4 component Weyl spinor with 8 real degrees of freedom. Therefore, for $\mathcal{D}=5,6$ the most simple supersymmetric theories, those with one copy of the supersymmetry generators ($\mathcal{N}=1$), will have the same field content as a $\mathcal{D}=4$ $\mathcal{N}=2$ theory when dimensionally reduced.

2.1 Free hypermultiplets

The superfield formulation of the $\mathcal{D}=5$ hypermultiplet has been described in [11]. In $\mathcal{N}=1,\mathcal{D}=4$ superspace, the 5D hypermultiplet consists of a collection of 4D chiral superfields $H(x_5), H^c(x_5)$ labeled by the 5'th co-ordinate x_5. Its free action is given by

$$S_5^{\text{Hyp.}}=\int d^5x\left\{\int d^4\theta\left(\bar{H}^cH^c+\bar{H}H\right)+\left(\int d^2\theta H^c\left(\partial_5+m\right)H+\text{h.c.}\right)\right\}. \tag{2.1}$$

Expanding in components and integrating out the auxilliary F components, the action (2.1) describes an $\mathcal{N}=1$ $\mathcal{D}=5$ supersymmetric theory containing two complex scalar and one Dirac fermion $\Psi_5^T=(\psi,\bar{\psi}^c)$ composed of the 2 component fermions ψ and $\bar{\psi}^c$:

$$\begin{aligned}S_5^{\text{Hyp.}}&=-\partial_MH^\dagger\partial^MH-\partial_MH^{c\dagger}\partial^MH^c-i\bar{\psi}\bar{\sigma}^m\partial_m\psi-i\bar{\psi}^c\bar{\sigma}^m\partial_m\psi^c-\psi^c\partial_5\psi-\bar{\psi}^c\partial_5\bar{\psi}-\\&\quad-m^2(|H^c|^2+|H|^2)-m(\psi^c\psi+\bar{\psi}^c\bar{\psi})\\&=-\partial_MH^\dagger\partial^MH-\partial_MH^{c\dagger}\partial^MH^c-m^2(|H^c|^2+|H|^2)+\bar{\Psi}_5(i\gamma^M\partial_M-m)\Psi_5\,.\end{aligned} \tag{2.2}$$

Here and throughout the paper, the capitalized indices run over $0,1,2,3,5$ while the lower-case ones run over $0,1,2,3$

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In $\mathcal{D}=6$ there are only Weyl and Dirac fermions, so the smallest multiplet contains a left or right Weyl fermion. The action for massless hypermultiplets is.

$$\begin{aligned} S_6^{\text{L. Hyp.}} &= \int d^6x \left\{ \int d^4\theta \left(\bar{H}_L^c H_L^c + \bar{H}_L H_L \right) + \left(\int d^2\theta H_L^c \, \partial H_L + \text{h.c.} \right) \right\} \\ S_6^{\text{R. Hyp.}} &= \int d^6x \left\{ \int d^4\theta \left(\bar{H}_R^c H_R^c + \bar{H}_R H_R \right) + \left(\int d^2\theta H_R^c \, \bar{\partial} H_R + \text{h.c.} \right) \right\} \end{aligned} \tag{2.3}$$

with

$$\begin{aligned} z &= \frac{1}{2}(x_5 + ix_6) & \bar{z} &= \frac{1}{2}(x_5 - ix_6) \\ \partial &= \frac{\partial}{\partial z} = \partial_5 - i\partial_6 & \bar{\partial} &= \frac{\partial}{\partial \bar{z}} = \partial_5 + i\partial_6 \,. \end{aligned} \tag{2.4}$$

Note that the rotational invariance of the transverse 2 dimensional space is realized as $z \to e^{i\theta} z, H_L^{(c)} \to e^{i\theta/2} H_L^{(c)}, H_R^{(c)} \to e^{-i\theta/2} H_R^{(c)}$. Therefore, unlike the 5D case, we can not make a massive hypermultiplet out of just e.g. H_L, H_L^c. Instead we must combine a copy of each of the massless hypermultiplets.

$$\begin{aligned} S_6^{\text{Hyp. Massive}} = \int d^6x \Bigg\{ & \int d^4\theta \left(\bar{H}_L^c H_L^c + \bar{H}_L H_L + \bar{H}_R^c H_R^c + \bar{H}_R H_R \right) + \\ & + \left(\int d^2\theta \begin{pmatrix} H_R^c & H_L^c \end{pmatrix} \begin{pmatrix} m & \bar{\partial} \\ \partial & m \end{pmatrix} \begin{pmatrix} H_L \\ H_R \end{pmatrix} + \text{h.c.} \right) \Bigg\}. \end{aligned} \tag{2.5}$$

The Dirac spinor is now

$$\Psi_6^{\text{Dirac}} = \begin{pmatrix} \Psi_6^{\text{Left Weyl}} \\ \Psi_6^{\text{Right Weyl}} \end{pmatrix}. \tag{2.6}$$

Note that the superpotential term is just the Dirac operator in the transverse 2D space.

2.2 Abelian gauge theory

The first step in formulating the higher dimensional theories in terms of ordinary $\mathcal{D}=4$, $\mathcal{N}=1$ superspace is to identify the correct superfields for the theory. The $\mathcal{D}=5$ super Yang-Mills theory will have a 5-vector gauge field, a 4 component Dirac gaugino, and a scalar. When dimensionally reduced down to $\mathcal{D}=4$, the gauge field becomes a 4-vector and a scalar, the gaugino splits into two Majorana gauginos, and the scalar is unaffected. So we must have a vector multiplet and chiral multiplet. This is also obvious since there are 8 real supercharges in 5D, which translates to $\mathcal{N}=2$ SUSY in 4D, with the $\mathcal{N}=2$ vector multiplet composed on an $\mathcal{N}=1$ vector and chiral multiplet.

The correct identification of the fields inside the vector field $V(x_5)$ and chiral field $\phi(x_5)$ is (with V in the Wess-Zumino gauge, and ϕ in the y-basis)

$$\begin{aligned} V &= -\theta\sigma^m\bar{\theta} A_m + i\bar{\theta}^2\theta\lambda_1 - i\theta^2\bar{\theta}\bar{\lambda}_1 + \frac{1}{2}\bar{\theta}^2\theta^2 D \\ \phi &= \frac{1}{\sqrt{2}}\left(\Sigma + iA_5\right) + \sqrt{2}\theta\lambda_2 + \theta^2 F \,. \end{aligned} \tag{2.7}$$

In the above and for the rest of the paper, the dependence of the 4D superfields on the extra co-ordinates is implicit. We also demand full $\mathcal{D} = 5$ gauge invariance of the theory. The gauge transformations of these 2 superfields are:

$$\begin{aligned} V &\to V + \Lambda + \bar{\Lambda} \\ \phi &\to \phi + \sqrt{2}\partial_5\Lambda\,. \end{aligned} \tag{2.8}$$

The subset of these transformations that preserve the Wess-Zumino gauge, correspond exactly to the ordinary $\mathcal{D} = 5$ gauge transformations.

It is easy to find a gauge invariant action

$$S_5^{\mathrm{A}} = \int d^5x \left[\frac{1}{4g^2}\int d^2\theta\; W^\alpha W_\alpha + \text{h.c.} + \int d^4\theta \frac{1}{g^2}\left(\partial_5 V - \frac{1}{\sqrt{2}}(\phi+\bar{\phi})\right)^2\right] \tag{2.9}$$

The first term is familiar and obviously gauge invariant; the second term is also clearly invariant under gauge tranformations with the variation of $\partial_5 V$ being canceled by that of $\phi + \bar{\phi}$.

While the $\mathcal{N} = 1$ SUSY is manifest in this lagrangian, the full SUSY and higher-dimensional Lorentz invariance is not. To see this, we expand the lagrangian in components in Wess-Zumino gauge. Keeping only the Bosonic fields the lagrangian becomes

$$\begin{aligned} &-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2g^2}D^2 \\ &-\frac{1}{2g^2}\partial_5 A_\mu \partial_5 A^\mu + \frac{1}{g^2}\partial_5 A_\mu \partial^\mu A_5 - \frac{1}{g^2}\Sigma\partial_5 D \\ &-\frac{1}{2g^2}(\partial_\mu\Sigma\partial^\mu\Sigma + \partial_\mu A_5 \partial^\mu A_5)\,. \end{aligned} \tag{2.10}$$

We can integrate out the auxiliary field D by setting it to its equation of motion, which is

$$D = -\partial_5\Sigma\,. \tag{2.11}$$

The lagrangian then naturally arranges itself into the form

$$-\frac{1}{4g^2}F_{MN}F^{MN} - \frac{1}{2g^2}\partial_M\Sigma\partial^M\Sigma \tag{2.12}$$

which is precisely the bosonic part of the 5D vector multiplet composed of the vector field A_M and the real scalar Σ. The full lagrangian including the fermions clearly also works out correctly.

This superspace form of the 5D lagrangian is very simple but does not straightforwardly generalize to higher dimensions, because $\phi, \bar{\phi}$ will transform oppositely under rotations in the transverse space. We can however re-write the action as

$$S_5^{\mathrm{A}} = \int d^5x \left[\frac{1}{4g^2}\int d^2\theta W^\alpha W_\alpha + \text{h.c.} + \int d^4\theta \frac{1}{g^2}\left((\sqrt{2}\partial_5 V - \bar{\phi})(\sqrt{2}\partial_5 V - \phi) - \partial_5 V \partial_5 V\right)\right]. \tag{2.13}$$

The gauge invariance of the second and third terms are not as manifest in this form, but it is easy to check. Under a gauge transformation,

$$\int d^4\theta(\sqrt{2}\partial_5 V-\bar{\phi})(\sqrt{2}\partial_5 V-\phi) \to \int d^4\theta(\sqrt{2}\partial_5 V-\bar{\phi}+\sqrt{2}\partial_5\Lambda)(\sqrt{2}\partial_5 V-\phi+\sqrt{2}\partial_5\bar{\Lambda})=$$
$$= \int d^4\theta(\sqrt{2}\partial_5 V-\bar{\phi})(\sqrt{2}\partial_5 V-\phi) + \int d^4\theta\left[2\partial_5 V\partial_5(\Lambda+\bar{\Lambda})+2\partial_5\Lambda\partial_5\bar{\Lambda}\right], \qquad (2.14)$$

where we have used the fact that purely chiral or anti-chiral terms vanish under the full superspace integration. Similarly,

$$-\int d^4\theta(\partial_5 V)^2 \to -\int d^4\theta(\partial_5 V+\partial_5\Lambda+\partial_5\bar{\Lambda})^2 =$$
$$= -\int d^4\theta(\partial_5 V)^2 - \int d^4\theta\left[2\partial_5 V\partial_5(\Lambda+\bar{\Lambda})-2\partial_5\Lambda\partial_5\bar{\Lambda}\right] \qquad (2.15)$$

so that the sum of the last two terms in eqn. (2.13) is gauge invariant.

The extension to $\mathcal{D}=6$ is simple. The transverse rotational invariance is useful as a guide to constructing the action. z transforms as $z \to e^{i\theta}z$, and we suspect that ϕ will combine with ∂ to form a covariant derivative so we define ϕ to transform as $\phi \to e^{-i\theta}\phi$. V is neutral. The gauge transformations are

$$\begin{aligned} V &\to V+\Lambda+\bar{\Lambda} \\ \phi &\to \phi+\sqrt{2}\partial\Lambda\,. \end{aligned} \qquad (2.16)$$

The 6D action is then the obvious extension of the 5D one:

$$S_6^{\mathrm{A}} = \int d^6x\left[+\frac{1}{4g^2}\int d^2\theta W^\alpha W_\alpha + \text{h.c.} + \int d^4\theta\frac{1}{g^2}\left((\sqrt{2}\bar{\partial}V-\bar{\phi})(\sqrt{2}\partial V-\phi)-\partial V\bar{\partial}V\right)\right]. \qquad (2.17)$$

In this case the lowest component of the superfield ϕ is

$$\phi\,|_{\theta=\bar{\theta}=0} = \frac{1}{\sqrt{2}}A = \frac{1}{\sqrt{2}}\left(A_6+iA_5\right). \qquad (2.18)$$

This expression reproduces the $\mathcal{D}=5$ super Yang-Mills action when all dependence on x_6 is eliminated and identifying A_6 as the scalar, Σ, of the $\mathcal{D}=5$ super Yang-Mills theory. The auxiliary field D is now proportional to F_{56}:

$$D = -\frac{1}{2}\left(\partial\bar{A}+\bar{\partial}A\right) = -\left(\partial_5 A_6-\partial_6 A_5\right) = -F_{56}\,. \qquad (2.19)$$

2.3 Non-abelian theory

We now generalize to the case of a non-abelian theory. Since ϕ contains the components of the higher dimensional gauge field, it must transform in the adjoint. With the definitions:

$$h = e^{-\Lambda} \qquad \bar{h} = e^{-\bar{\Lambda}} \qquad (2.20)$$

the gauge transforms become

$$\phi \to h^{-1}(\phi-\sqrt{2}\partial)h \qquad e^V \to h^{-1}e^V\bar{h}^{-1}, \qquad (2.21)$$

where $\phi \equiv \phi^a T^a$ and $V \equiv V^a T^a$.

The natural guess for the non-abelian action would be to simply insert various factors of e^V:

$$S_6^{\mathrm{NA}} = \int d^6x \Bigg\{ \frac{1}{4kg^2} \,\mathrm{Tr}\left[\int d^2\theta\, W^\alpha W_\alpha + \mathrm{h.c.}\right] + \\ + \int d^4\theta\, \frac{1}{kg^2}\,\mathrm{Tr}\left[(\sqrt{2}\bar\partial + \bar\phi)e^{-V}(-\sqrt{2}\partial + \phi)e^V + \bar\partial e^{-V}\partial e^V\right]\Bigg\}\,, \quad (2.22)$$

where $\mathrm{Tr}T^aT^b = k\delta^{ab}$. This action reproduces the $\mathcal{D} = 6$ non-abelian super Yang-Mills theory in Wess-Zumino gauge. However, this action is not fully gauge invariant under the gauge transformation. As pointed out in [12], we need to add one more term to make it perfectly gauge invariant:

$$\int d^6x \int d^4\theta \frac{1}{kg^2}\,\mathrm{Tr}\left[\bar\partial V \frac{\sinh L_V - L_V}{L_V^2}\partial V\right]. \quad (2.23)$$

This has the structure of a WZW term. We refer to [12] for details on the variation of this term. Here we note that the term is absent in $\mathcal{D} = 5$ and in all cases, vanishes in Wess-Zumino gauge. Therefore, one can use (2.22), together with any desired couplings to brane fields, and obtain the correct lagrangian in Wess-Zumino gauge. To find the $\mathcal{D} = 5$ non-abelian theory, one removes the x_6 dependence. This action also reproduces the appropriate abelian theories when all fields commute.

The auxiliary field, D, in $\mathcal{D} = 6$ again becomes the higher dimensional field strength, only this time it is the non-abelian field strength F_{56}:

$$D = -\frac{1}{2}\left(\bar\partial A + \partial\bar A + [A, \bar A]\right) = -F_{56}\,. \quad (2.24)$$

We can dimensionally reduce this to $\mathcal{D} = 5$ to get

$$D = -\left(\partial_5\Sigma + i[\Sigma, A_5]\right) = -D_5\Sigma\,, \quad (2.25)$$

with D_5 being x_5 component of the covariant derivative.

2.4 Coupling to hypermultiplets

It is easy to extend our action for free hypermultiplets to the case where they are charged under a gauge symmetry. With the hypermultiplets belonging to a representation R of the gauge group G, we have the gauge transformations:

$$H \to hH \\ H^c \to h^{c-1}H^c\,, \quad (2.26)$$

with $h = e^{-\Lambda^a T^a_R}$ and $h^c = (h^{-1})^T = (e^{\Lambda^a T^a_R})^T$. The generalization of our previous hypermultiplet action is trivial; we simply replace the ordinary ∂_5 derivatives with the covariant derivative $\partial_5 - \frac{1}{\sqrt{2}}\phi$:

$$S_5^{\text{Hyp. Gauge}} = \int d^5x \Bigg\{ \int d^4\theta [H^c e^V \bar H^c + \bar H e^{-V} H] + \\ + \left[\int d^2\theta \left(H^c\left(m + \left(\partial_5 - \frac{1}{\sqrt{2}}\phi\right)\right)H\right) + \ \mathrm{h.c.}\right]\Bigg\} \quad (2.27)$$

(of course here $V = V^a T^a_R, \phi = \phi^a T^a_R$).

In $\mathcal{D}=6$ we must choose our gaugino to be either a left or right handed Weyl field. To make a covariant derivative we must combine the left-handed ϕ with ∂ and the right-handed field with $\bar{\partial}$. Therefore, hypermultiplets of a given handedness can not couple to gauge fields of opposite handedness. The action for a $\mathcal{D}=6$ hypermultiplet coupled to a gauge field of the same handedness is

$$S_6^{\text{Hyp. Gauge}} = \int d^6x \left\{ \int d^4\theta [H^c e^V \bar{H}^c + \bar{H} e^{-V} H] + \right. \\ \left. + \left[\int d^2\theta \, \sqrt{2} H^c (\partial - \frac{1}{\sqrt{2}}\phi) H + \text{ h.c.} \right] \right\}. \tag{2.28}$$

3. $\mathcal{D}=7$ to 10

Spinors in $\mathcal{D}=7$ to 10 dimensions have a minimum of 16 real components. This means that there are a minimum of 16 real supercharges and thus all theories in these dimensions must be constructed out of $\mathcal{N}=4$ multiplet in the 4D language. The $\mathcal{N}=4$ vector multiplet decomposes under $\mathcal{N}=1$ as 3 chiral multiplets ϕ_i and a vector multiplet V, so we need to build our superspace lagrangian out of these fields.

We will only consider the $\mathcal{D}=10$ theory because it is easy to dimensionally reduce to $\mathcal{D}=7$ to 9. It will be convenient to use complex coordinates,z^i, for the transverse space with

$$z^1 = \frac{1}{2}(x_5 + ix_6) \qquad z^2 = \frac{1}{2}(x_7 + ix_8) \qquad z^3 = \frac{1}{2}(x_9 + ix_{10}) \,. \tag{3.1}$$

The transverse rotational invariance is the SO(6) rotating the $x_5, \ldots, x_{10}$ into each other. The SU(3) subgroup rotating the z_i will be useful in constructing invariant actions. We will use the convention that $\bar{z}_i = (z^i)^\dagger$, and that $\bar{\phi}^i = (\phi_i)^\dagger$.

Again we will find that the higher dimensional components of the gauge field will be the lowest components of ϕ_i:

$$\phi_j \,|_{\theta=\bar{\theta}=0} = \frac{1}{\sqrt{2}} A_j = \frac{1}{\sqrt{2}}(A_{4+2j} + iA_{3+2j}) \\ j \in \{1,2,3\} \,. \tag{3.2}$$

This choice of the embedding was to make the dimensional reduction from $\mathcal{D}=10$ most transparent.

3.1 Abelian theory

The appropriate gauge transformation for this theory are

$$V \to V + (\Lambda + \bar{\Lambda}) \\ \phi_i \to \phi_i + \sqrt{2}\partial_i \Lambda \,.$$

The Kähler potential of the theory is the natural generalization of the $\mathcal{D}=5,6$ theory. However, this will not reproduce the correct theory. We need to introduce a superpotential that will complete the gauge potential kinetic term. This is also obvious since, if we reduce the theory to 6D eliminating the dependence on $x_7, \ldots, x_{10}$, ϕ_2, ϕ_3 form a hypermultiplet in 6D, and as we have seen the hyper-multiplet kinetic term is completed by a superpotential term. In any case, the SU(3) symmetry and the known result when reduced to $D = 6$ specifies everything, and we have for the action

$$\begin{aligned} S_{10}^{\mathrm{A}} = \int d^{10}x \Bigg\{ \int d^2\theta \left(\frac{1}{4g^2} W^\alpha W_\alpha + \frac{1}{2g^2} \epsilon^{ijk} \phi_i \partial_j \phi_k \right) + \text{h.c.} + \\ + \int d^4\theta \frac{1}{g^2} \left[\left(\sqrt{2}\partial_i V - \phi_i \right) \left(\sqrt{2}\bar{\partial}^i V - \bar{\phi}^i \right) - \partial_i V \bar{\partial}^i V \right] \Bigg\} . \end{aligned} \tag{3.3}$$

Note that the gauge variation of the superpotential vanishes via integration by parts and the antisymmetry of ϵ^{ijk}. The auxiliary fields F_i, D are given by

$$\begin{aligned} D &= -\frac{1}{2}(\partial_i \bar{A}^i + \bar{\partial}^i A_i) \\ F^{i\dagger} &= -\frac{1}{\sqrt{2}} \epsilon^{ijk} \partial_j A_k \, . \end{aligned} \tag{3.4}$$

3.2 Non-abelian theory

The non-abelian action is the natural generalization of the abelian one. The superpotential must be modified to make it gauge invariant, which is accomplished by replacing the $\partial_j \phi_k$ with $\partial_j \phi_k - [\phi_j, \phi_k]/3\sqrt{2}$. The gauge transformations are

$$\begin{aligned} \phi_i &\to h^{-1}(\phi_i - \sqrt{2}\partial_i) h e^V \to h^{-1} e^V \bar{h}^{-1} \\ S_{10}^{\mathrm{NA}} &= \int d^{10}x \Bigg\{ \int d^2\theta \ \mathrm{Tr} \left(\frac{1}{4kg^2} W^\alpha W_\alpha + \frac{1}{2kg^2} \epsilon^{ijk} \phi_i (\partial_j \phi_k - \frac{1}{3\sqrt{2}} [\phi_j, \phi_k]) \right) + \\ &\quad + \int d^4\theta \ \frac{1}{kg^2} \ \mathrm{Tr} \left((\sqrt{2}\bar{\partial}^i + \bar{\phi}^i) e^{-V} (-\sqrt{2}\partial_i + \phi_i) e^V + \bar{\partial}^i e^{-V} \partial_i e^V \right) \Bigg\} + \\ &\quad + \text{WZW term} \, . \end{aligned} \tag{3.5}$$

Once again, the last term vanishes in W-Z gauge. Note that the superpotential has the structure of a Chern-Simons term. Under a gauge transformation, the superpotential transforms as

$$\begin{aligned} \mathrm{Tr}\epsilon^{ijk} \phi_i \left(\partial_j \phi_k + \frac{1}{\sqrt{2}} [\phi_j, \phi_k] \right) \to \mathrm{Tr}\epsilon^{ijk} \phi_i \left(\partial_j \phi_k + \frac{1}{\sqrt{2}} [\phi_j, \phi_k] \right) - \\ -2\sqrt{2}\mathrm{Tr} \left[\epsilon^{ijk} (\partial_i h) h^{-1} (\partial_j h) h^{-1} (\partial_k h) h^{-1} \right] . \end{aligned} \tag{3.6}$$

The last term is a total derivative and is the Pontryagin density. For the transformations that preserve WZ gauge, $\Lambda = \exp(i\theta\sigma^m\bar{\theta}\partial_m) a(x)$ with no higher components, the Pontryagin term vanishes identically under the superspace integration.

Finally some brief comments on the R symmetry of these theories. In $\mathcal{D} = 10$, the transverse rotational symmetry is SO(6) which is homomorphic to SU(4). This SU(4)

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symmetry is the R symmetry of the $\mathcal{D} = 4$ $\mathcal{N} = 4$ theory. The superpotential and Kähler terms we have written have only an explicit SU(3) symmetry. This SU(3) is a subgroup of the $\mathrm{SU}(4)_R$, keeping $\mathcal{N} = 1$ SUSY manifest. By writing our theory in terms of $\mathcal{N} = 1$ superfields, we choose a special supersymmetry generator and break the manifest SU(4) R symmetry. We maintain an SU(3) subgroup which transforms the three supersymmetry generators that are orthogonal to our $\mathcal{N} = 1$ SUSY generator.

4. Some applications

4.1 Coupling to sources

In [5], the "shining" of bulk massive fields by sources localized on branes was considered as a mechanism for producing small parameters on the brane. This is a consequence of the exponentially small profile for the massive field in the bulk, which is given by the massive Yukawa propagator. It is natural to try and extend this mechanism to supersymmetric theories. This was done in [11] for the case of 5D theories, as we review below. A source was added to a massive bulk hypermultiplet of the form

$$\int d^2\theta dx_5 \delta(x_5) J H^c \,. \tag{4.1}$$

The F-flatness conditions become:

$$\begin{aligned} -F^\dagger &= (m - \partial_5) H^c = 0 \\ -F^{c\dagger} &= J\delta(x_5) + (m + \partial_5) H = 0 \,. \end{aligned} \tag{4.2}$$

These equations have solution $H^c = 0$ and:

$$H = -\theta(x_5) J e^{-my} \tag{4.3}$$

in infinite space and

$$H = \frac{-Je^{-my}}{1 - e^{-2\pi m R}} \tag{4.4}$$

on a circle of radius R.

We can do a similar thing for the free hypermultiplets in 6D . We add:

$$\int d^6x d^2\theta \delta(x_5)\delta(x_6) J H_L^c \,. \tag{4.5}$$

The F flatness conditions are:

$$F_L^\dagger = \partial H_L^c + m H_R^c = 0 \tag{4.6}$$
$$F_L^{c\dagger} = -\partial H_L + m H_R - J\delta(z\bar{z}) = 0 \tag{4.7}$$
$$F_R^{c\dagger} = -\bar{\partial} H_R + m H_L = 0 \tag{4.8}$$
$$F_R{}^\dagger = \bar{\partial} H_R^c + m H_L^c = 0 \,. \tag{4.9}$$

Consider first the massless case $m = 0$ and $H_R^c = H_L^c = H_R = 0$; then we have

$$\partial H_L = -J\delta(z\bar{z}) \tag{4.10}$$

which has solution:

$$H_L = -J\frac{\theta(z\bar{z})}{\bar{z}} = -J\frac{\theta(x_5^2+x_6^2)}{x_5-ix_6} = -J\frac{e^{i\phi}}{r}\theta(r^2) \tag{4.11}$$

in infinite space. In order to find the solution on a compact space, say a torus, we could use the method of images.

In the massive case, we take $H^c_{L,R}=0$ and combine equations (4.7) and (4.8) to get:

$$\begin{aligned}\bar{\partial}\partial H_R - m^2 H_R &= -mJ\delta^2(z\bar{z})\\ H_L &= \frac{1}{m}\bar{\partial}H_R\,.\end{aligned} \tag{4.12}$$

The first equation, is just the Klein-Gordon equation in 2D, so the solution is the Yukawa potential in 2D. For large mr, we have

$$H_R \sim -Jme^{-mr}, H_L = -Jme^{-mr}e^{i\phi}\,. \tag{4.13}$$

It is interesting that H_L acquires a "vortex" profile in the transverse two dimensions. Even if all the parameters in the lagrangian are real, this vortex profile breaks CP. If the Standard Model Yukawa couplings arise through shining via branes that do not all fall on a straight line, the phase in H_L can be used to introduce CP violation into the SM in an amusing way.

4.2 Charged matter on branes

Using our formalism, it is very easy to couple bulk gauge fields to charged matter on boundaries. We simply add the following term to the appropriate higher-dimensional action:

$$\int d^4x d^4\theta \bar{X} e^{-V}|_{z=0} X\,, \tag{4.14}$$

where X is a 4 dimensional chiral superfield living on a brane and z represents the extra dimensions. For example, in 5D abelian case, the action would be:

$$\begin{aligned}&\text{5D free action} + \text{4D free action} +\\ &+\frac{1}{g^2}\int d^4x\Bigg[A^n\left[-\frac{1}{2}\bar{\lambda}_X\bar{\sigma}^n\lambda_X - \frac{i}{2}\bar{A}_X\partial_n A_X\right] + i\frac{1}{\sqrt{2}}\left(A_X\bar{\lambda}_X\bar{\lambda}_1 + \text{h.c}\right) -\\ &\qquad -\frac{1}{4}A_nA^n\bar{A}_XA_X\frac{1}{2}D\bar{A}_XA_X\Bigg]\,,\end{aligned} \tag{4.15}$$

where A_X is the scalar component of the X multiplet. These are just the usual couplings of a 4D chiral superfield with a 4D vector superfield. But, in our case, the D term is different. Let's examine the D part of the lagrangian in detail:

$$\mathcal{L}_D = \frac{1}{g^2}\left(\frac{1}{2}D^2 + D\partial_5\Sigma - \frac{1}{2}\bar{A}_XA_XD\delta(x_5)\right)\,. \tag{4.16}$$

The first two terms come from the free 5D action part. Upon eliminating D we get

$$
\begin{aligned}
\mathcal{L}_D &= -\frac{1}{2g^2}\left(\partial_5\Sigma - \frac{1}{2}\bar{A}_X A_X \delta(x_5)\right)^2 \\
&= -\frac{1}{2g^2}\left((\partial_5\Sigma)^2 - \partial_5\Sigma\bar{A}_X A_X\delta(x_5) + \frac{1}{4}(\bar{A}_X A_X)^2\delta(0)\delta(x_5)\right). \qquad (4.17)
\end{aligned}
$$

This result was obtained earlier in [10], but our derivation makes the ease of the superspace formalism transparent. It is also trivial to extend the result to higher dimensions, for instance in 6D we have

$$
\begin{aligned}
\mathcal{L}_D &= -\frac{1}{2g^2}\left(F_{56} - \frac{1}{2}\bar{A}_X A_X \delta(z)\right)^2 \\
&= -\frac{1}{2g^2}\left(F_{56}^2 - F_{56}\bar{A}_X A_X\delta^2(z) + \frac{1}{4}(\bar{A}_X A_X)^2\delta^2(0)\delta^2(z)\right), \qquad (4.18)
\end{aligned}
$$

where $F_{56} = (\partial_5 A_6 - \partial_6 A_5)$. We note that if $F_{56}(z=0) \neq 0$ then we get SUSY breaking soft scalar masses proportional to the strength of the magnetic field on the brane.

4.3 Orbifolds

Our formalism is also useful for constructing field-theoretic orbifolds [13] preserving $\mathcal{N}=1$ SUSY in 4D. Such constructions are useful both for obtaining chiral fermions as well as reduced supersymmetry in the low-energy 4D theory.

The simplest canonical example is the $S_1/\mathbb{Z}_2$ orbifold in the 5D case [14]. Consider as an example a U(1) gauge theory in 5D. It is trivial to see that our 5D action is invariant under

$$
\begin{aligned}
V(x_\mu, x_5) &\to V(x_\mu, -x_5) \\
\phi(x_\mu, x_5) &\to -\phi(x_\mu, -x_5)\,. \qquad (4.19)
\end{aligned}
$$

To construct the orbifolded model, we only keep states that are invariant under the symmetry, as well as periodic under $x_5 \to x_5 + 2L$. That is we impose

$$
\begin{aligned}
V(x_\mu, x_5) &= V(x_\mu, -x_5) \\
\phi(x_\mu, x_5) &= -\phi(x_\mu, -x_5) \qquad (4.20)
\end{aligned}
$$

as well as

$$
\begin{aligned}
V(x_\mu, x_5 + 2L) &= V(x_\mu, x_5) \\
\phi(x_\mu, x_5 + 2L) &= \phi(x_\mu, x_5)\,. \qquad (4.21)
\end{aligned}
$$

The physical space is then the interval $[0, L]$. In order to obtain the low-energy theory, we only need to look at x_5 independent modes that satisfy the above boundary conditions. Evidently, we get a zero mode from V but not from ϕ, and so the low-energy theory is pure 4D, $\mathcal{N}=1$ U(1) theory.

In 6D, we can e.g. compactify on $T^2/\mathbb{Z}_3$, by imposing ($\omega^3 = 1$)

$$\begin{aligned} V(x_\mu, z, \bar{z}) &= V(x_\mu, \omega z, \bar{\omega}\bar{z}) \\ \phi(x_\mu, z, \bar{z}) &= \omega\phi(x_\mu, \omega z, \bar{\omega}\bar{z}) \end{aligned} \tag{4.22}$$

together with the periodicity conditions on the torus. Again, $\phi = 0$ at the fixed points of the torus, and this projects out the theory to $\mathcal{N} = 1$ SYM in 4D.

The non-abelian case offers more interesting possibilities, since we can combine a gauge transformation with the orbifold symmetry. Of course, for chiral theories in even dimensions, we need to worry about anomaly cancellation in the bulk. A simple anomaly-free example in say $D = 6$ is obtained, however, by imagining that we dimensionally reduce from e.g. seven dimensions where there are no anomalies. The 6D particle content is then a vector multiplet and a hypermultiplet in the adjoint representation. The 6D pure gauge anomaly is proportional to

$$\mathrm{tr}F^4_{\mathrm{Adj.}} - \mathrm{tr}F^4_{\mathrm{Hyper}} \tag{4.23}$$

which clearly vanishes for a simple hyper in the adjoint rep. (The gravitational anomalies can be canceled with the Green-Schwarz mechanism.)

For an amusing example, suppose we start with an SU(9) theory in 6D. We will again compactify on $T^2/\mathbb{Z}_3$, but this time using the orbifold symmetry

$$\begin{aligned} V(x_\mu, z, \bar{z}) &= U^\dagger V(x_\mu, \omega z, \bar{\omega}\bar{z})U \\ \phi_i(x_\mu, z, \bar{z}) &= \omega U^\dagger \phi_i(x_\mu, \omega z, \bar{\omega}\bar{z})U\,. \end{aligned} \tag{4.24}$$

Where U is a 9×9 matrix written in term of 3×3 blocks:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{4.25}$$

V, ϕ can also be written as a general 9×9 matrix:

$$V = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}, \qquad \phi = \begin{pmatrix} A' & B' & C' \\ D' & E' & F' \\ G' & H' & I' \end{pmatrix}. \tag{4.26}$$

Now,

$$U^\dagger V U = \begin{pmatrix} A & \omega B & \omega^2 C \\ \omega^2 D & E & \omega F \\ \omega G & \omega^2 H & I \end{pmatrix}, \qquad \omega U^\dagger \phi U = \begin{pmatrix} \omega A' & \omega^2 B' & C' \\ D' & \omega E' & \omega^2 F' \\ \omega^2 G' & H' & \omega I' \end{pmatrix}. \tag{4.27}$$

We see that for the zero modes of V, only (A, E, I) survive. This means that the low-energy theory is $\mathcal{N} = 1$ with gauge group $\mathrm{SU}(3)^3$. On the other hand, from the ϕ_i, (C', D', H') survive, which transform under $\mathrm{SU}(3)^3$ as $\psi_i \sim (\bar{3}, 3, 1,), (3, 1, \bar{3}), (1, \bar{3}, 3)$. There is also a superpotential coupling $\psi_1\psi_2\psi_3$ which is inherited from the $H^c\phi H$ superpotential term. This model is just the particle content of "trinification", with 3 generations, and a single large Yukawa coupling.

5. Localizing chiral fermions

It is well-known that it is possible to localize chiral Fermions on defects in extra dimensions [15]. The simplest example are Fermions localized to domain walls. Consider a Fermion in 5 dimensions with a spatially varying mass term $m(x_5)$ (which could for instance arise from a Yukawa coupling to a scalar field with a "kink" profile in the 5'th dimension). It is easy to see that if $m(+\infty) > 0$ and $m(-\infty) < 0$, then upon solving for the spectrum of the Dirac operator we find a chiral zero mode with wavefunction peaked around the location where the mass term goes through zero. Of course, if we attempt to compactify the fifth dimension on a circle, then we necessarily have a kink and an anti-kink, and we don't get a chiral spectrum in the 4D theory. However, we can combine Fermion localization with an orbifold to keep the e.g. the left-handed localized zero mode but project out the right handed one, as in [16].

In this section we will supersymmetrize these models and address some physical questions that arise. Consider 5D theories. Note that charged hyper-multiplets in the bulk have a superpotential coupling $H^c\phi H$, which is an effective mass term when ϕ is non-zero. If we can arrange for ϕ to vary and change sign from one side of a brane to another, then we can localize one of H, H^c to the brane. It is easy to arrange for this to happen. The simplest example to consider is a 5D theory with a U(1) gauge field in the bulk, and a brane located at $x_5 = 0$. We will add a Fayet-Iliopoulos term for the gauge bulk gauge field on the brane. The action is

$$\text{Free 5D action} + \int d^4x \int d^4\theta 2\zeta V(x, x_5 = 0)\,. \tag{5.1}$$

The D term is now given by

$$D = -\partial_5\Sigma + 2\zeta\delta(x_5)\,. \tag{5.2}$$

The most general solution to the $D-$flatness conditions is then

$$\Sigma(x_5) = \Sigma_0 + \zeta \mathrm{sgn}(x_5)\,. \tag{5.3}$$

This is a "kink" for Σ. There is moduli space of vacua labeled by Σ_0. Note that in the range $|\Sigma_0| < |\zeta|$, $\Sigma(x_5)$ changes sign as it goes through the origin, while for $|\Sigma_0| > |\zeta|$, $\Sigma(x_5)$ is non-vanishing and of the same sign everywhere.

Now, let us add a bulk Hypermuliplet with charge $+1$ under the U(1). Treating the gauge field as a background,the hypermultiplet action becomes

$$\int dx_5 \int d^4\theta \bar{H}H + \bar{H}^c H^c + \int d^2\theta H^c\left(\partial_5 - \frac{1}{2}\Sigma(x_5)\right)H + \text{h.c.} \tag{5.4}$$

If we are to have zero modes for H or H^c, their wavefunctions $\psi(x_5), \psi^c(x_5)$ must satisfy

$$\begin{aligned}\left(\partial_5 - \frac{1}{2}\Sigma(x_5)\right)\psi &= 0\\ \left(\partial_5 + \frac{1}{2}\Sigma(x_5)\right)\psi^c &= 0\,.\end{aligned} \tag{5.5}$$

The solutions are trivially

$$\begin{aligned}\psi(x_5) &= \psi(0)e^{\int_0^{x_5} dy\frac{1}{2}\Sigma(y)} = \psi(0)e^{\frac{1}{2}(\Sigma_0+\zeta\text{sgn}(x_5))x_5}\\ \psi^c(x_5) &= \psi^c(0)e^{-\int_0^{x_5} dy\frac{1}{2}\Sigma(y)} = \psi^c(0)e^{-\frac{1}{2}(\Sigma_0+\zeta\text{sgn}(x_5))x_5}\,.\end{aligned} \tag{5.6}$$

Clearly for $|\Sigma_0| > |\zeta|$, neither of these solutions is normalizable and there are no localized chiral zero modes. However, for $|\Sigma_0| < |\zeta|$, one (but not the other) of the above two wavefunctions will be normalizable and we localize a chiral fermion to the brane at $x_5 = 0$.

So, in one region of moduli space $|\Sigma_0| < |\zeta|$, we have a chiral zero mode but for $|\Sigma_0| > |\zeta|$ it disappears. How can the net chirality change as we smoothly move around in moduli space? Mathematically, as $|\Sigma_0| \to |\zeta|$, the wavefunction of the chiral zero mode spreads out more and more till at $|\Sigma_0| = |\zeta|$ it it is unnormalizable. Physically, what is going on is also transparent. For $|\Sigma_0| < |\zeta|$, there is a normalizable zero mode, and then (since the Fermions are massive both for $x_5 > 0$ and $x_5 < 0$) there is mass gap above which we have the full 5D continuum. Therefore the low-energy theory is indeed 4-dimensional, and remains that way as we smoothly vary Σ_0. However, as $|\Sigma_0|$ approaches $|\zeta|$, the bulk Fermion mass on one side of the brane approaches zero till exactly at $|\Sigma_0| = \zeta$, the bulk mass term is zero on one side and the low-energy theory is not 4D. As we continue to $|\Sigma_0| > |\zeta|$, the bulk Fermion become massive in the bulk again. But to an observer on the brane, net chirality has been changed. There is of course no contradiction with the usual statement that net chirality cannot change in 4D, because as we move around in moduli space we go through a region where there is no effective 4D description. This is an elementary analog of the chirality-changing transitions in string theory discussed in [17]. There, chirality changing transitions occured while moving around in moduli space, when the effective 4D *field theory* description of the physics broke down due to the appearance of tensionless strings.

There are simple variations on the above model. For instance, instead of introducing a FI term on the brane, we could introduce a pair of chiral fields $X, \bar{X}$ of charge $+1, -1$, which can take arbitrary vevs. Normally, this would violate $D-$ flatness, but in this case we simply have

$$D = -\partial_5\Sigma + (|X|^2 - |\bar{X}|^2)\delta(x_5) \tag{5.7}$$

and so we have have the same $D-$flat solution for Σ as before with $\zeta \to (|X|^2 - |\bar{X}|^2)$.

We can also easily discuss compactification and the supersymmetric generalization of the models in [16]. We will consider an $S_1/\mathbb{Z}_2$ of the model with a U(1) gauge field and a hypermultiplet in the bulk. The orbifold symmetry is

$$\begin{aligned}V(x,x_5) &\to V(x,-x_5)\\ \phi(x,x_5) &\to -\phi(x,-x_5)\\ H(x,x_5) &\to H(x,-x_5)\\ H^c(x,-x_5) &\to -H^c(x,-x_5)\,.\end{aligned} \tag{5.8}$$

Furthermore, on the orbifold fixed point at $x_5 = 0$ we will write down a FI term, while on the fixed point at $x_5 = L$ we will put a pair of chiral fields $Y, \bar{Y}$ of charge $+1, -1$. The $D-$

flatness condition is now

$$\partial_5\Sigma + 2\zeta\delta(x_5) + (|Y|^2 - |\bar{Y}|^2)\delta(x_5 - L) = 0\,. \tag{5.9}$$

The general solution to this equation is

$$\Sigma(x_5) = \Sigma_0 + \zeta\mathrm{sgn}(x_5) + \frac{1}{2}(|\chi|^2 - |\bar{\chi}|^2)\mathrm{sgn}(x_5 - L)\,. \tag{5.10}$$

However, in order to be able to find a solution invariant under the orbifold symmetry we must have

$$\Sigma_0 = 0\,, \qquad |\chi|^2 - |\bar{\chi}|^2 + 2\zeta = 0\,. \tag{5.11}$$

So the Σ modulus has been projected out, and the second condition is just the usual D-flatness condition in 4D, (as it had to be from the low-energy point of view).

Now, we can look at what happens to the hypermultiplets in this background. The solutions for the zero mode wavefunctions of H, H^c are

$$\psi = Ae^{\frac{\zeta}{2}x_5}\,, \qquad \psi^c = Be^{-\frac{\zeta}{2}x_5} \tag{5.12}$$

however, by the orbifold symmetry, the zero mode of H^c must vanish at the orbifold fixed points, so B must vanish and there is therefore no zero mode for H^c. The zero mode for H, on the other hand, can be localized at either $x_5 = 0$ or $x_5 = L$ depending on the sign of ζ.

Finally, we can replace the FI term on the fixed point at $x_5 = 0$ by another pair of chiral multiplets $X, \bar{X}$ of charge ± 1. Then everything goes through the same with $\zeta \to (|X|^2 - |\bar{X}|^2)$. There is a moduli space of solutions corresponding to the usual $D-$flat space in 4D. As we move along this moduli space, the chiral fermion wavefunction can shift from being localized around $x_5 = 0$, to having a flat wavefunction, to being localized around $x_5 = L$.

6. Anomalies and super-Chern-Simons theory

We have shown how to localize chiral fermions in a fifth direction. It is then natural to ask what happens with anomalies in such a theory. In the case of domain-wall fermions, it is well-known that the apparent anomaly due to the localized chiral zero mode is canceled by the variation of a Chern-Simons term in the bulk [18]. We will not repeat the whole story here. The important point is that the variation of the Chern-Simons action on a manifold M, under a gauge transformation $\delta A = d\Lambda$, is

$$\delta\int_M A\wedge F\wedge F = \int_M d\Lambda\wedge F\wedge F = \int_M d(\Lambda F\wedge F) = \int_{\partial M}\Lambda F\wedge F \tag{6.1}$$

and the integral over the boundary has precisely the form of a 4D anomaly. As such, it can cacel the 4D anomaly induced by fields living at the boundaries.

What we would like to do here is show how to supersymmetrize the 5D Chern-Simons term in our 4D superspace formalism. Note that the usual Chern-Simons term contains

$A_5F\tilde{F}$. It is easy to see that this term must come from the term $\int d^2\theta\phi WW$. However, note that the gauge coupling of the 5D YM theory can be absorbed by shifting ϕ. This leads us to guess that 5D SYM + 5D super-Chern-Simons theory is actually on the moduli space of pure 5D super-Chern-Simons. As we will see, this is indeed correct. We will therefore only construct the action for pure super-Chern-Simons theory.

The piece of the action we have so far, $\int d^2\theta\phi WW$, is clearly not fully gauge invariant, nor fully 5D Lorentz-invariant. It is not difficult to find the correct combination of terms required. The correct action for super-Chern-Simons theory, on an interval between $x_5 = [y_1, y_2]$, is

$$\begin{aligned} S^{\rm 5DCS} &= \int d^4x \int_{y_1}^{y_2} dx_5 \int d^2\theta\phi WW + \text{h.c} - \\ &-\frac{\sqrt{2}}{3}\int d^4\theta(\partial_5 V D_\alpha V W^\alpha - V D_\alpha \partial_5 V W^\alpha) + \text{h.c} - \\ &-\frac{1}{3}\int d^4\theta(\sqrt{2}\partial_5 V - (\phi+\bar{\phi}))^3 \,. \end{aligned} \tag{6.2}$$

The bosonic part of the component action is:

$$S^{\rm 5DCS}_{\rm bosonic} = \int d^4x \int_{y_1}^{y_2} dx_5 - \frac{1}{2\sqrt{2}}\epsilon^{MNOPQ}A_M F_{NO} F_{PQ} + \frac{1}{\sqrt{2}}\Sigma F^{MN}F_{MN} + \sqrt{2}\Sigma\partial_M\Sigma\partial^M\Sigma \,. \tag{6.3}$$

It is easy to verify with this action that under the gauge transformation $\delta\phi = \partial_5\Lambda, \delta V = \Lambda + \bar{\Lambda}$, the above action has a variation

$$\delta S^{\rm 5DCS} = \int d^4x \int d^2\theta(\Lambda WW)(y_2) - (\Lambda WW)(y_1) + \text{h.c} \tag{6.4}$$

which is the full supermultiplet of chiral anomalies on the boundaries at y_1, y_2.

7. Conclusions

In this paper we have given the rules for constructing globally supersymmetric lagrangians from $\mathcal{D} = 5$ to $\mathcal{D} = 10$ dimensions in the familiar $\mathcal{N} = 1, \mathcal{D} = 4$ superspace. This makes it easy to do explicit supersymmetric model-building in extra dimensions, in particular allowing us to couple bulk fields to fields localized on 3-branes with ease. We illustrated the utility of the formalism with a number of simple examples. It would be interesting to explore some generalizations of these examples in detail. For instance, it would be nice to generalize the supersymmetric localization of chiral fermions to higher dimensions.

There are also a number of possible extensions of the ideas in this paper that we have not touched on. For instance, when we have dynamical branes which fluctuate, with finite tension, there are massless scalar fields living on the branes which are the goldstone bosons of spontaneously broken translational invariance. They non-linearly realize the full translational symmetry of the theory [19]. In the case where the brane preserves some SUSY, these goldstone modes must fall into supermultiplets, and it would be interesting to know how to systematically construct lagrangians non-linearly realizing the full SUSY. A related possible application of our formalism is the construction of BPS ($\mathcal{N}=1$ preserving) solitons in the higher-dimensional theory. These would be $F-$ and $D-$ flat solutions with non-trivial variation of fields in the extra dimensions. Finally, it would of be desirable to extend our formalism to the case of supergravity. While a full treatment may be difficult, the case of linearized supergravity could be tractable, and would already contain much of the interesting physics.

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